

APPROXIMATING THE SPECTRUM OF MATRICES AND HYPERMATRICES

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ABSTRACT. We describe a new method for computing generators of elimination ideals associated with matrix and hypermatrix spectral constraints. We proceed to derive from these generators iterative procedures for approximating the spectral decomposition of matrices and hypermatrices.

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1. INTRODUCTION

Many combinatorial optimization problems including instances of subgraph-isomorphism. These combinatorial problems are known to be NP-hard, and often one seeks to identify special families of graphs for which efficient combinatorial algorithms can be devised. The complexity of ensuing algorithms is often determined by combinatorial and algebraic properties of graph encodings. A good illustration of this fact is provided by the graph property of being an expander graph. The complexity of many combinatorial algorithms for such a graph can be expressed in terms of its expansion parameter [AC88, Alo86, ASS08, AM85, BL06, Che70, Chu97, Li01]. As is well known, many graph properties, including the property of being an expander graph, are closely tied to the spectral decomposition of matrices deduced from incidence structures in the associated graph [Alo86, Che70, Chu97, ST11]. The well known graph adjacency matrix is constructed such that the (i_1, i_2) matrix entry equals 1 if the associated graph admits a directed edge connecting vertex i_1 to vertex i_2 , and equals 0 otherwise. Following a construction proposed

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in [FW95] by Friedman and Widgerson, one also associates with a graph a $(k-1)$ -*path adjacency hypermatrix*. The $(k-1)$ -path adjacency hypermatrix corresponds to a k -th order hypermatrix (note that ordinary matrices are second order hypermatrices) whose $(i_1, i_2 \dots, i_k)$ entry equals 1 if the ordered tuple $(i_1, i_2 \dots, i_k)$ denotes a directed path of length $k-1$ in the associated graph, and equals 0 otherwise. Consequently, graph algebraic invariants derived from the corresponding adjacency matrix are extended to include polynomial relations between entries of path adjacency hypermatrices. The polynomial relations which are retained as algebraic and combinatorial invariants are usually the ones which are invariant under the natural action of the permutation group. Such algebraic relations are to be thought of as generalizations of the classical Cayley-Hamilton matrix polynomial. Just as it is done for matrices, the algebraic varieties defined by the algebraic and combinatorial invariants will be referred to as the *spectrum* or the *spectral decomposition*. The invariants associated with path adjacency hypermatrices enable us to distinguish some non-isomorphic graphs with isospectral adjacency matrices.

The spectral analysis of hypermatrices is considerably more difficult to define [Chu93, Lub14, PRT12, FGL⁺11, GER11, CD13, QSW13, LSQ14, FW95, Qi12, IGZ94] when compared with the spectral analysis of matrices. Mesner and Bhattacharya in [MB90, MB94] introduced an m operands product for m -th order hypermatrices. E. Gnan, V. Retakh and A. Elgammal proposed in [GER11] a generalization to hypermatrices of the notions of Hermiticity and unitarity. E. Gnan, V. Retakh and A. Elgammal also proved in [GER11] a conjecture of Bhattacharya by using these new definitions to extend the spectral decomposition to hypermatrices of arbitrary order. In the present work we show that the spectral decomposition introduced in [GER11] for a hypermatrix is mostly determined by the spectral decomposition its minors in a similar spirit to Cauchy's interlacing theorem. We show in the present work that the majority of hypermatrix product proposed in the literature including the Segre outer product, the contraction product, multilinear matrix multiplication [Lim13], as well as the Kerner product [Ker08] are either special cases of the general BM product or special cases of the dual product to the general BM product. We present new algorithms for deriving generators of the elimination ideals associated with matrix and hypermatrix spectral decomposition constraints. The proposed algorithms are based on generalization of Parsevals' identities also known as resolution of the identity. We derive from the generators the spectral elimination ideal iterative procedures for approximating the spectral decomposition of matrices and hypermatrices. We extend to even order hypermatrices the self-adjoint argument for establishing the existence of real solutions to spectral constraints. Finally we deduce from the the spectral decomposition of hypermatrices upper and lower bounds for hypermatrix eigenvalues introduced by L.H. Lim and L.Q. Qi in [Lim05, Qi05].

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2. NOTATION

We describe for convenience of the reader the notation used throughout the paper. The Hadamard product of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m \times n}$ noted $\mathbf{A} \circ \mathbf{B}$, yields a matrix of the same

dimensions whose entries are the product of corresponding entries of \mathbf{A} and \mathbf{B} ,

$$(i, j) - \text{th entry of } (\mathbf{A} \circ \mathbf{B}) \text{ is } a_{i,j} b_{i,j}.$$

The vector product of $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{n \times 1}$ with the background $n \times n$ matrix \mathbf{M} refers to the bilinear form associated with \mathbf{M} expressed as

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{M}} := \sum_{0 \leq k_0, k_1 < n} a_{k_0} m_{k_0, k_1} b_{k_1},$$

in particular it follows that

$$\langle \mathbf{a}, \mathbf{b} \rangle := \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{I}_n} = \sum_{0 \leq k < n} a_k b_k$$

where the entries of \mathbf{I}_n are given by

$$[\mathbf{I}_n]_{i,j} := \left(\delta_{i,j} = \begin{cases} 1 & \text{if } 0 \leq i = j < n \\ 0 & \text{otherwise} \end{cases} \right).$$

The inner-product of $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{n \times 1}$ is $\langle \mathbf{a}, \bar{\mathbf{b}} \rangle$. For $\{\mathbf{v}_j\}_{0 \leq j < n} \subset \mathbb{C}^{n \times 1}$ we define the correlation product noted $\langle \mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1} \rangle$ to be

$$\langle \mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1} \rangle := \langle \mathbf{1}_{n \times 1}, \mathbf{v}_0 \circ \dots \circ \mathbf{v}_{n-1} \rangle.$$

It shall also be convenient to adopt the notation convention

$$\mathbf{a}^{\circ^\alpha} := ((a_k)^\alpha)_{0 \leq k < n}.$$

The n -dimensional vector \mathbf{w} denotes the vector collecting powers of the primitive n -th roots of unity with entries given by

$$\mathbf{w} := (w_j = \exp(2\pi i j/n))_{0 \leq j < n}.$$

Finally, we associate with an arbitrary $\mathbf{v} \in \mathbb{C}^{n \times 1}$ the $n \times n$ Vandermonde matrix

$$[\text{Vandermonde}(\mathbf{v})]_{i,j} = (v_j)^i.$$

3. OVERVIEW OF THE BHATTACHARYA-MESNER ALGEBRA AND ITS DUAL

We recall here for convenience of the reader the basic elements of the Bhattacharya-Mesner (BM) algebra proposed in [MB90, MB94] as a generalization of the algebra of matrices.

Definition 1. The Bhattacharya-Mesner [MB90, MB94] algebra generalizes the classical matrix product

$$\mathbf{B} = \mathbf{A}^{(1)} \cdot \mathbf{A}^{(2)}$$

where $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{B}$ are matrices of sizes $n_1 \times k, k \times n_2, n_1 \times n_2$, respectively,

$$b_{i_1, i_2} = \sum_{1 \leq j \leq k} a_{i_1, j}^{(1)} a_{j, i_2}^{(2)},$$

to an m -operand hypermatrix product noted

$$\mathbf{B} = \text{Prod}(\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(m)}),$$

where \mathbf{B} is an $n_1 \times \cdots \times n_m$ hypermatrix, for $i = 1, \dots, (m-1)$, $\mathbf{A}^{(i)}$ is a hypermatrix whose size is obtained by replacing n_{i+1} by k in the dimensions of the hypermatrix \mathbf{B} , and $\mathbf{A}^{(m)}$ is a $k \times n_2 \times \cdots \times n_m$ hypermatrix, and

$$b_{i_1, \dots, i_m} = \sum_{1 \leq j \leq k} a_{i_1, \mathbf{j}, i_3, \dots, i_m}^{(1)} \cdots a_{i_1, \dots, i_t, \mathbf{j}, i_{t+2}, \dots, i_m}^{(t)} \cdots a_{\mathbf{j}, i_2, \dots, i_m}^{(m)}.$$

In the particular case of third order hypermatrix product noted

$$\mathbf{B} = \text{Prod} \left(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)} \right)$$

where $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$, $\mathbf{A}^{(3)}$ and \mathbf{B} are hypermatrices of sizes $n_1 \times k \times n_3$, $n_1 \times n_2 \times k$, $k \times n_2 \times n_3$ and $n_1 \times n_2 \times n_3$ respectively,

$$b_{i_1, i_2, i_3} = \sum_{1 \leq j \leq k} a_{i_1, \mathbf{j}, i_2}^{(1)} a_{i_1, i_2, \mathbf{j}}^{(2)} a_{\mathbf{j}, i_1, i_2}^{(3)}.$$

A slight variation of the BM product was introduced in [GER11]. The proposed variation of the BM product is called the general BM product and noted

$$\mathbf{C} = \text{Prod}_{\mathbf{B}} \left(\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(m)} \right).$$

The resulting hypermatrix \mathbf{C} is an $n_1 \times \cdots \times n_m$ hypermatrix, while the dimensions of the hypermatrix $\mathbf{A}^{(i)}$ for $i = 1, \dots, m-1$ is obtained by replacing n_{i+1} by k in the dimensions of \mathbf{C} and $\mathbf{A}^{(m)}$ is a hypermatrix of size $k \times n_2 \times \cdots \times n_m$ similarly to the BM product. Crucially, the general BM product differs from the BM product in the fact that the general BM product involves an additional input hypermatrix. The additional input hypermatrix \mathbf{B} is called the background hypermatrix and as such \mathbf{B} must be a cubic m -th order hypermatrix having all of its sides of length k i.e. \mathbf{B} is of dimension $k \times k \times \cdots \times k$,

$$c_{i_1, \dots, i_m} = \sum_{1 \leq j_1, \dots, j_m \leq k} a_{i_1, \mathbf{j}_2, i_3, \dots, i_m}^{(1)} \cdots a_{i_1, \dots, i_t, \mathbf{j}_{t+1}, i_{t+2}, \dots, i_m}^{(t)} \cdots a_{\mathbf{j}_1, i_2, \dots, i_m}^{(m)} b_{j_1, \dots, j_m}.$$

Note that the original BM product is recovered by setting \mathbf{B} to the Kronecker delta hypermatrix (i.e. the hypermatrix whose nonzero entries all equal one and are located at the entries whose indices all have the same value, in particular Kronecker delta matrices are identity matrices).

Consider the product

$$\mathbf{D} = \mathbf{C} \circ \text{Prod}_{\mathbf{B}} \left(\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(m)} \right),$$

hence

$$d_{i_1, \dots, i_m} = c_{i_1, \dots, i_m} \sum_{1 \leq j_1, \dots, j_m \leq k} a_{i_1, \mathbf{j}_2, i_3, \dots, i_m}^{(1)} \cdots a_{i_1, \dots, i_t, \mathbf{j}_{t+1}, i_{t+2}, \dots, i_m}^{(t)} \cdots a_{\mathbf{j}_1, i_2, \dots, i_m}^{(m)} b_{j_1, \dots, j_m}.$$

We define the dual to the general BM product to be expressed as

$$d_{\mathbf{j}_1, \dots, \mathbf{j}_m} = b_{j_1, \dots, j_m} \sum_{1 \leq i_1, \dots, i_m \leq k} a_{i_1, \mathbf{j}_2, i_3, \dots, i_m}^{(1)} \cdots a_{i_1, \dots, i_t, \mathbf{j}_{t+1}, i_{t+2}, \dots, i_m}^{(t)} \cdots a_{\mathbf{j}_1, i_2, \dots, i_m}^{(m)} c_{i_1, \dots, i_m}. \quad (3.1)$$

The duality here arises from interchanging the indices in the summands with the indices of the hypermatrix \mathbf{C} in the product. Note that in the case of matrices, the product is self dual. The product dual to the general BM product was independently

proposed by Kerner in [Ker08]. We also note that most hypermatrix product in the literature including the Segre outer product, the contraction product, the multilinear matrix multiplication [Lim13] all correspond to special instances of the general BM product with additional constraints imposed on the input hypermatrices.

4. SPECTRAL DECOMPOSITION FROM THE SPECTRUM OF MINORS.

We discuss here a general argument for reducing the spectral decomposition of an arbitrary cubic k -th order hypermatrix to the decomposition of cubic minors of same order having sides of length k . The hypermatrices considered here are associated with directed and weighted k -uniform hypergraph having no degenerate edges. In other words, the collection of k vertices which make up any hyperedge of the hypergraph must be distinct. Such hypermatrices arise as $(k-1)$ -path adjacency hypermatrices of rooted trees whose edges are directed towards the leaf nodes and away from the root. Such hypermatrices also arise as $(k-1)$ -path adjacency hypermatrices of directed acyclic graphs.

Theorem 2. *Let H denote a directed weighted k -uniform hypergraph having no degenerate hyperedges. Then the spectral decomposition of its k -vertex sub-hypergraphs determine the spectral decomposition of a larger k -uniform hypergraph on $n \binom{n}{k}$ vertices which admit H as a sub-hypergraph.*

We will first present the detail proof in the case of graphs and subsequently extend the arguments to hypergraphs.

Proof. Theorem 2 asserts that for a directed and weighted graph having no loop edges noted

$$G_1 := (V_1 = \{0, 1, \dots, n-1\}, E_1 \subset V_1 \times V_1),$$

the spectral decomposition of its two vertex subgraph adjacency matrices determine the spectral decomposition of the adjacency matrix of a larger graph

$$G_2 := \left(V_2 = \left\{ 0, 1, \dots, n \binom{n}{2} - 1 \right\}, E_2 \subset V_2 \times V_2 \right)$$

which admit G_1 as a subgraph. Let \mathbf{A} denote the $n \times n$ adjacency matrix of the graph G_2 . We seek to determine the spectral decomposition of the adjacency matrix of G_2 expressed as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}_{01} \\ \mathbf{B}_{10} & \mathbf{B}_{11} \end{pmatrix} = (\mathbf{U} \cdot \text{diag}\{\boldsymbol{\mu}\}) \cdot (\mathbf{V} \cdot \text{diag}\{\boldsymbol{\nu}\})^\dagger, \quad \mathbf{U} \cdot \mathbf{V}^\dagger = \mathbf{I}_{n \binom{n}{2}}$$

where the sub matrices \mathbf{B}_{01} , \mathbf{B}_{10} and \mathbf{B}_{11} are matrices of size respectively given by $n \times (n-1) \binom{n}{2}$, $(n-1) \binom{n}{2} \times n$, and $(n-1) \binom{n}{2} \times (n-1) \binom{n}{2}$. Incidentally the matrices \mathbf{U} and \mathbf{V} denotes $n \binom{n}{2} \times n \binom{n}{2}$ matrix which are respectively associated with basis for the left and right eigenspaces respectively. Finally the vectors $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ are $n \binom{n}{2}$ dimensional vectors such that the entries of their Hadamard product $\boldsymbol{\mu} \star \boldsymbol{\nu}$ yield the eigenvalues of the adjacency matrix of G_2 . The k -th column vector of the matrices $\mathbf{U} \cdot \text{diag}\{\boldsymbol{\mu}\}$ and $\mathbf{V} \cdot \text{diag}\{\boldsymbol{\nu}\}$ denote the scaled left eigenvectors and right scaled eigenvectors respectively. For $0 \leq j_1 < j_2 < n$ let the matrix $\mathbf{A}^{[j_0, j_1]}$ denote $n \times n$ matrices defined by

$$\mathbf{A}^{[j_1, j_2]} = \mathbf{A} \circ \sum_{\sigma \in S_2} \mathbf{e}_{\sigma(j_1)} \otimes \mathbf{e}_{\sigma(j_2)}^T$$

where the set $\{\mathbf{e}_j\}_{0 \leq j < n}$ denotes column vectors of the identity matrix \mathbf{I}_n . By construction we have

$$\mathbf{A} = \sum_{0 \leq j_1 < j_2 < n} \mathbf{A}^{[j_1, j_2]}.$$

Furthermore let the spectral decomposition of the matrix $\mathbf{A}^{[j_1, j_2]}$ be expressed as

$$\begin{aligned} \mathbf{A}^{[j_1, j_2]} &= \left(\mathbf{U}^{[j_1, j_2]} \cdot \text{diag} \left\{ \boldsymbol{\mu}^{[j_1, j_2]} \right\} \right) \cdot \left(\mathbf{V}^{[j_1, j_2]} \cdot \text{diag} \left\{ \boldsymbol{\nu}^{[j_1, j_2]} \right\} \right)^\dagger \\ \mathbf{e}_{j_1} \cdot \mathbf{e}_{j_1}^T + \mathbf{e}_{j_2} \cdot \mathbf{e}_{j_2}^T &= \mathbf{U}^{[j_1, j_2]} \cdot \left(\mathbf{V}^{[j_1, j_2]} \right)^\dagger. \end{aligned}$$

we have

$$\begin{aligned} \forall 0 \leq i_1, i_2 < n, \quad a_{i_1, i_2} = \\ \sum_{\substack{0 \leq k < n \\ 0 \leq j_1 < j_2 < n}} \left[\left(\sqrt{n-1} \mu_k^{[j_1, j_2]} \right) \left(\frac{u_{i_0, k}^{[j_1, j_2]}}{\sqrt{n-1}} \right) \right] \overline{\left[\left(\sqrt{n-1} \nu_k^{[j_1, j_2]} \right) \left(\frac{v_{i_1, k}^{[j_1, j_2]}}{\sqrt{n-1}} \right) \right]}, \end{aligned} \quad (4.1)$$

and

$$\forall 0 \leq i_1, i_2 < n, \quad \delta_{i_1, i_2} = \sum_{\substack{0 \leq k < n \\ 0 \leq j_1 < j_2 < n}} \left(\frac{u_{i_0, k}^{[j_1, j_2]}}{\sqrt{n-1}} \right) \overline{\left(\frac{v_{i_1, k}^{[j_1, j_2]}}{\sqrt{n-1}} \right)}.$$

The right-hand side of the expressions in 4.1 should be viewed as expressing inner-products of $n \binom{n}{2}$ -dimensional vectors. The first n rows of the matrices \mathbf{U} and $\bar{\mathbf{V}}$ are therefore determined by the expansion above. The remaining $\left(\binom{n}{2} - 1 \right) n$ rows of the matrix \mathbf{U} and $\bar{\mathbf{V}}$ are determined by the Gram-Schmidt process.

The proposed construction used for adjacency matrices of graphs is easily extended to higher order hypermatrices via the BM algebra. For notational convenience we discuss here only the third order hypermatrice case. We recall from [GER11] that by analogy to the matrix case the spectral decomposition a third order hypermatrix \mathbf{A} is expressed in terms of scaled eigenmatrices as follows

$$\mathbf{A} = \text{Prod} \left(\text{Prod} \left(\mathbf{Q}, \mathbf{D}_3, \mathbf{D}_3^T \right), \left[\text{Prod} \left(\mathbf{U}, \mathbf{D}_2, \mathbf{D}_2^T \right) \right]^{T^2}, \left[\text{Prod} \left(\mathbf{V}, \mathbf{D}_1, \mathbf{D}_1^T \right) \right]^T \right).$$

The collection of row-depth matrix slices of the hypermatrices \mathbf{Q} , \mathbf{U} and \mathbf{V} yields bases for the *eigenmatrices* of \mathbf{A} which are also subject to the non-correlation constraints

$$\left[\text{Prod} \left(\mathbf{Q}, \mathbf{U}^{T^2}, \mathbf{V}^T \right) \right]_{i_1, i_2, i_2} = \begin{cases} 1 & \text{if } i_1 = i_2 = i_3 \\ 0 & \text{otherwise} \end{cases}.$$

The scaling hypermatrices $\{\mathbf{D}_i\}_{1 \leq i \leq 3}$ correspond to third order hypermatrix analog of diagonal matrices. Consequently, the row-depth slices of the hypermatrices $\text{Prod}(\mathbf{Q}, \mathbf{D}_3, \mathbf{D}_3^T)$, $\text{Prod}(\mathbf{U}, \mathbf{D}_2, \mathbf{D}_2^T)$, and $\text{Prod}(\mathbf{V}, \mathbf{D}_1, \mathbf{D}_1^T)$, correspond to scaled eigenmatrices of \mathbf{A} . By analogy to the matrix case, for all triplets label (j_1, j_2, j_3) for which

the inequality $0 \leq j_1 < j_2 < j_3 < n$ is satisfied, we define the matrix $\mathbf{A}^{[j_1, j_2, j_3]}$ to be the third order hypermatrix expressed by

$$\mathbf{A}^{[j_1, j_2, j_3]} = \mathbf{A} \circ \sum_{\sigma \in S_3} \mathbf{e}_{\sigma(j_1)} \otimes \mathbf{e}_{\sigma(j_2)}^T \otimes \mathbf{e}_{\sigma(j_3)}^{T^2}. \quad (4.2)$$

where the set $\{\mathbf{e}_j\}_{0 \leq j < n}$ denotes column vectors of the identity matrix \mathbf{I}_n and the transpose operation here refers to the cyclic permutation of the hypermatrix entries as introduced in [GER11]. Similarly we have

$$\mathbf{A} = \sum_{0 \leq j_1 < j_2 < j_3 < n} \mathbf{A}^{[j_1, j_2, j_3]} \quad (4.3)$$

by appropriately concatenating the spectral decomposition of the hypermatrix minors $\mathbf{A}^{[j_0, j_1, j_2]}$ and using the generalization to hypermatrices of the constrained inverse matrix pair problem, we deduce the spectral decomposition of a larger $n \binom{n}{3} \times n \binom{n}{3} \times n \binom{n}{3}$ which admits the hypermatrix \mathbf{A} as a sub-hypermatrix.

More generally a similar construction is easily devised for higher order hypermatrices. The argument therefore provides a way to reduce the spectral decomposition of k -th order cubic hypermatrices to the spectral decomposition of cubic sub-hypermatrices with sides of length k . \square

5. COMPUTING SPECTRAL ELIMINATION IDEALS.

For the purpose of introducing new iterative procedures to approximate the spectral decomposition of matrices and hypermatrices, We describe here two elimination procedures for computing generators of elimination ideals associated with matrix and hypermatrix spectral constraints. The first elimination method uses only properties of the algebra of matrices to obtain generators for the ideal associated with eigenvalues in the matrix case and scaling values in the general case of higher order hypermatrices. The first elimination method also allows us to derive new generalizations of the matrix determinant polynomial for higher order hypermatrices. The second elimination method uses a Hypermatrix generalization of the classical Paserval identity to obtain generators for the ideal associated with entries of uncorrelated tuples (which are hypermatrix analog of matrix eigenvectors). The iterative procedure described subsequently will use the generators devised by the second method.

5.1. Spectral elimination ideals for matrices. We start by discussing both elimination method in the matrix case and subsequently proceed to extend the arguments to hypermatrices of arbitrary orders. Let us recall for convenience of the reader the well known matrix spectral decomposition of an $n \times n$ matrix \mathbf{A} . Such a decomposition is obtained by solving for $n \times n$ matrices \mathbf{U} , \mathbf{V} , and diagonal matrices $\{\mathbf{D}_i\}_{1 \leq i \leq 2}$ subject to the spectral constraints

$$\begin{cases} \mathbf{A} &= (\mathbf{U} \cdot \mathbf{D}_1) \cdot (\overline{\mathbf{V} \cdot \mathbf{D}_2})^T \\ \mathbf{U} \cdot \overline{\mathbf{V}}^T &= \mathbf{I}_n \\ \mathbf{D}_i \circ \mathbf{D}_i &= \mathbf{D}_i^T \cdot \mathbf{D}_i, \quad \forall 0 \leq i < 2 \end{cases}. \quad (5.1)$$

For notational convenience we reformulate the diagonality constraints in the spectral decomposition, in terms of n -dimensional vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ as follows

$$\begin{cases} \mathbf{A} &= (\mathbf{U} \cdot \text{diag}\{\boldsymbol{\mu}\}) \cdot (\mathbf{V} \cdot \text{diag}\{\boldsymbol{\nu}\})^\dagger \\ \mathbf{U} \cdot \mathbf{V}^\dagger &= \mathbf{I}_n \end{cases}.$$

The two elimination ideals obtained by the two distinct method derived from the spectral decomposition constraints are :

$$\mathcal{I}_{\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}} := \mathbb{C}[\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}] \cap \text{Ideal generated by } \left\{ (\mathbf{U} \cdot \text{diag}\{\boldsymbol{\mu}\}) \cdot (\mathbf{V} \cdot \text{diag}\{\boldsymbol{\nu}\})^\dagger - \mathbf{A}, \mathbf{U} \cdot \mathbf{V}^\dagger - \mathbf{I}_n \right\},$$

and

$$\mathcal{I}_{\mathbf{U}, \overline{\mathbf{V}}} := \mathbb{C}[\mathbf{U}, \overline{\mathbf{V}}] \cap \text{Ideal generated by } \left\{ (\mathbf{U} \cdot \text{diag}\{\boldsymbol{\mu}\}) \cdot (\mathbf{V} \cdot \text{diag}\{\boldsymbol{\nu}\})^\dagger - \mathbf{A}, \mathbf{U} \cdot \mathbf{V}^\dagger - \mathbf{I}_n \right\}.$$

Let us start by describing the derivation of generators for $\mathcal{I}_{\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}}$. The starting point for the derivation is the matrix identity

$$\forall 0 \leq k \leq n, \quad \mathbf{A}^k = \left(\mathbf{U} \cdot \text{diag}\{\boldsymbol{\mu}^{\circ k}\} \right) \cdot \left(\mathbf{V} \cdot \text{diag}\{\boldsymbol{\nu}^{\circ k}\} \right)^\dagger.$$

Let $\{\mathbf{u}_i\}_{0 \leq i < n}$ and $\{\overline{\mathbf{v}}_j\}_{0 \leq j < n}$ denote respectively row vectors of the matrices \mathbf{U} and $\overline{\mathbf{V}}$, we reformulate these identities into the following Vandermonde type equalities of the form

$$\forall 0 \leq i, j < n, \quad \mathbf{u}_i \circ \overline{\mathbf{v}}_j = (\text{Vandermonde}\{\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}\})^{-1} \cdot \begin{pmatrix} [\mathbf{A}^0]_{i,j} \\ \vdots \\ [\mathbf{A}^{n-1}]_{i,j} \end{pmatrix}$$

the constraints can be combined to yield the following identity.

$$\begin{pmatrix} [\mathbf{u}_i \circ \overline{\mathbf{v}}_j]_0 \\ \vdots \\ [\mathbf{u}_i \circ \overline{\mathbf{v}}_j]_{n-1} \end{pmatrix}_{0 \leq i, j < n} = (\mathbf{I}_{n^2} \otimes \text{Vandermonde}(\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}))^{-1} \cdot \begin{pmatrix} [\mathbf{A}^0]_{i,j} \\ \vdots \\ [\mathbf{A}^{n-1}]_{i,j} \end{pmatrix}_{0 \leq i, j < n}.$$

It is implicitly assumed in the identity above that \mathbf{A} has distinct eigenvalues, otherwise the identity above is expressed using the Penrose matrix inverse instead as follows

$$\begin{pmatrix} [\mathbf{u}_i \circ \overline{\mathbf{v}}_j]_0 \\ \vdots \\ [\mathbf{u}_i \circ \overline{\mathbf{v}}_j]_{n-1} \end{pmatrix}_{0 \leq i, j < n} = (\mathbf{I}_{n^2} \otimes \text{Vandermonde}(\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}))^+ \cdot \begin{pmatrix} [\mathbf{A}^0]_{i,j} \\ \vdots \\ [\mathbf{A}^{n-1}]_{i,j} \end{pmatrix}_{0 \leq i, j < n}.$$

Having thus expressed in the equality above the entries of the vectors $\{\mathbf{u}_i \circ \overline{\mathbf{v}}_j\}_{0 \leq i, j < n}$ only in terms of the entries of \mathbf{A} and its eigenvalues, we derive the generators for $\mathcal{I}_{\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}}$ as prescribed by tautologies

$$\mathcal{I}_{\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}} = \text{Ideal generated by } \{(\mathbf{u}_i \circ \overline{\mathbf{v}}_j) \circ (\mathbf{u}_j \circ \overline{\mathbf{v}}_i) - (\mathbf{u}_i \circ \overline{\mathbf{v}}_i) \circ (\mathbf{u}_j \circ \overline{\mathbf{v}}_j)\}_{0 \leq i, j < n}.$$

Note that the computation of resultants or alternatively the computation of Groebner basis can also be used to compute generators for $\mathcal{I}_{\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}}$ starting from the spectral constraints in 5.1 as pointed out in [GER11]. However these approaches are less direct and considerably less efficient because one must identify for spectral constraints good monomial orderings. The elimination method also has the benefit of producing explicit expressions for the characteristic polynomial and determinant. We illustrate this by considering the case $n = 2$, where there generator of the ideal $\mathcal{I}_{\boldsymbol{\mu} \circ \overline{\boldsymbol{\nu}}}$ results in a single vector equality

$$(\mathbf{u}_0 \circ \overline{\mathbf{v}}_1) \circ (\mathbf{u}_1 \circ \overline{\mathbf{v}}_0) - (\mathbf{u}_0 \circ \overline{\mathbf{v}}_0) \circ (\mathbf{u}_1 \circ \overline{\mathbf{v}}_1) = \mathbf{0}_{2 \times 1}$$

and yields the equality

$$\begin{pmatrix} (\mu_1 \overline{\nu_1})^2 - (a_{0,0} + a_{1,1}) \mu_1 \overline{\nu_1} + (a_{0,0} a_{1,1} - a_{0,1} a_{1,0}) \\ (\mu_0 \overline{\nu_0})^2 - (a_{0,0} + a_{1,1}) \mu_0 \overline{\nu_0} + (a_{0,0} a_{1,1} - a_{0,1} a_{1,0}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let us now turn our attention to the derivation of generators for the elimination ideal $\mathcal{I}_{\mathbf{U}, \overline{\mathbf{V}}}$. The derivation is achieved by eliminating the variables associated with the eigenvalues determined by the entries of the vectors $\boldsymbol{\mu}$ and $\overline{\boldsymbol{\nu}}$ via Parseval's identity. For the purposes of the proposed elimination method we view the spectral constraints in 5.1 as a collection of n^2 inner product equalities of the form

$$\{\langle \mathbf{u}_{i_0} \circ \boldsymbol{\mu}, \overline{\boldsymbol{\nu} \circ \mathbf{v}_{i_1}} \rangle = a_{i_0, i_1}\}_{0 \leq i_0, i_1 < n},$$

where the sets $\{\mathbf{u}_i\}_{0 \leq i < n}$ and $\{\mathbf{v}_i\}_{0 \leq i < n}$ denote row vectors of the matrix \mathbf{U} and \mathbf{V} respectively. By Parseval's identity the n^2 constraints can be reformulated as

$$\left\{ \sum_{0 \leq k < n} \langle \mathbf{u}_{i_0} \circ \boldsymbol{\mu}, \overline{\boldsymbol{\nu} \circ \mathbf{v}_{i_1}} \rangle (\overline{\mathbf{v}_k})^T \cdot \mathbf{u}_k = a_{i,j} = \sum_{0 \leq j_0, j_1 < n} \mu_{j_0} \overline{\nu_{j_1}} f_{n \cdot j_0 + j_1, n \cdot i_0 + i_1}(\mathbf{U}, \overline{\mathbf{V}}) \right\}_{0 \leq i_0, i_1 < n} \quad (5.2)$$

where $\{f_{n \cdot j_0 + j_1, n \cdot i_0 + i_1}(\mathbf{U}, \overline{\mathbf{V}})\}_{0 \leq n \cdot j_0 + j_1, n \cdot i_0 + i_1 < n^2} \subset \mathbb{C}[\mathbf{U}, \overline{\mathbf{V}}]$ and expressed by

$$f_{n \cdot j_0 + j_1, n \cdot i_0 + i_1}(\mathbf{U}, \overline{\mathbf{V}}) = u_{i_0 j_0} \left(\sum_{0 \leq k < n} \overline{v_{j_0 k}} u_{j_1 k} \right) \overline{v_{i_1 j_1}}.$$

The constraints form a system of n^2 equations in the n^2 unknowns $\{\mu_i \overline{\nu_j}\}_{0 \leq i, j < n}$. We therefore express the polynomial constraints in 5.2 as follows

$$\begin{pmatrix} f_{0,0} & \cdots & f_{0,j} & \cdots & f_{0,(n^2-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{i,0} & \cdots & f_{i,j} & \cdots & f_{i,(n^2-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{(n^2-1),0} & \cdots & f_{(n^2-1),j} & \cdots & f_{(n^2-1),(n^2-1)} \end{pmatrix} \begin{pmatrix} \mu_0 \overline{\nu_0} \\ \vdots \\ \mu_i \overline{\nu_j} \\ \vdots \\ \mu_{n-1} \overline{\nu_{n-1}} \end{pmatrix} = \begin{pmatrix} a_{0,0} \\ \vdots \\ a_{i,j} \\ \vdots \\ a_{(n-1),(n-1)} \end{pmatrix}. \quad (5.3)$$

For any integer $0 \leq k < n^2$, let \mathbf{F}_k denote the $n^2 \times n^2$ matrix constructed by substituting the k -th column of the left hand side matrix \mathbf{F} above by the righthand side $n^2 \times 1$ vector made up by the entries of \mathbf{A} as prescribed by the classical Cramer's rule. The generators for $\mathcal{I}_{\mathbf{U}, \overline{\mathbf{V}}}$ are determined by rational function identities derived from tautologies similar to the tautologies used in the first elimination method.

$$\{(\mu_i \overline{\nu_i}) (\mu_j \overline{\nu_j}) = (\mu_i \overline{\nu_j}) (\mu_j \overline{\nu_i})\}_{0 \leq i < j < n}.$$

It follows from Cramer's rule that generators for the ideal $\mathcal{I}_{\mathbf{U}, \overline{\mathbf{V}}}$ are derive from the rational identities

$$\left\{ \frac{\det(\mathbf{F}_{n \cdot i + i} \cdot \mathbf{F}_{n \cdot j + j}) - \det(\mathbf{F}_{n \cdot i + j} \cdot \mathbf{F}_{n \cdot j + i})}{\det(\mathbf{F}^2)} = 0 \right\}_{0 \leq i < j < n}.$$

The elimination ideal $\mathcal{I}_{\mathbf{U}, \overline{\mathbf{V}}}$ is seldom used in the literature however it's analog is of crucial importance for approximating the spectrum of hypermatrices.

5.2. Spectral elimination ideals for hypermatrices. For notational convenience we restrict the discussion here to third order hypermatrices. We recall the spectral decomposition for third order hypermatrices is expressed by

$$\mathbf{A} = \text{Prod} \left(\text{Prod}(\mathbf{Q}, \mathbf{D}_3, \mathbf{D}_3^T), [\text{Prod}(\mathbf{U}, \mathbf{D}_2, \mathbf{D}_2^T)]^{T^2}, [\text{Prod}(\mathbf{V}, \mathbf{D}_1, \mathbf{D}_1^T)]^T \right).$$

The collection of matrix slices of the hypermatrices \mathbf{Q} , \mathbf{U} and \mathbf{V} collects the *eigen-matrices* of \mathbf{A} which are subject to the constraints

$$\left[\text{Prod}(\mathbf{Q}, \mathbf{U}^{T^2}, \mathbf{V}^T) \right]_{i_1, i_2, i_2} = \Delta = \begin{cases} 1 & \text{if } i_1 = i_2 = i_3 \\ 0 & \text{otherwise} \end{cases}.$$

The scaling hypermatrices $\{\mathbf{D}_i\}_{1 \leq i \leq 3}$ are hypermatrix analog of diagonal matrices. Third order hypermatrix diagonality constraints are similar to matrix diagonality constraints and expressed by

$$\forall 1 \leq i \leq 3, \quad \mathbf{D}_i \circ \mathbf{D}_i \circ \mathbf{D}_i = \text{Prod}(\mathbf{D}_i^T, \mathbf{D}_i^{T^2}, \mathbf{D}_i),$$

The slices of $\text{Prod}(\mathbf{Q}, \mathbf{D}_3, \mathbf{D}_3^T)$, $\text{Prod}(\mathbf{U}, \mathbf{D}_2, \mathbf{D}_2^T)$, and $\text{Prod}(\mathbf{V}, \mathbf{D}_1, \mathbf{D}_1^T)$, correspond to scaled eigenmatrices of \mathbf{A} . Moreover the spectral decomposition constraints can be rewritten in terms of inner-product constraints of the form

$$\forall 0 \leq i, j, k < 2, \quad a_{ijk} = \langle (\alpha_i \circ \mathbf{q}_{ik} \circ \alpha_k), (\beta_j \circ \mathbf{u}_{ji} \circ \beta_i), (\gamma_k \circ \mathbf{v}_{kj} \circ \gamma_j) \rangle$$

$$\langle \mathbf{q}_{ik}, \mathbf{u}_{ji}, \mathbf{v}_{kj} \rangle = \delta_{i,j,k} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases} \quad \forall 0 \leq i, j, k < n.$$

The derivation according to the first elimination method is best illustrated for cubic hypermatrices having side length equal to 2. In particular for $2 \times 2 \times 2$ the spectral decomposition yields the following Vandermonde type equalities generally expressed

$$\forall 0 \leq i, j, k < 2, \quad \mathbf{q}_{ik} \circ \mathbf{u}_{ji} \circ \mathbf{v}_{kj} =$$

$$(\text{Vandermonde} \{ (\alpha_i \circ \alpha_k) \circ (\beta_j \circ \beta_i) \circ (\gamma_k \circ \gamma_j) \})^{-1} \begin{pmatrix} \delta_{i,j,k} \\ a_{i,j,k} \end{pmatrix}$$

Having thus expressed in the equalities above the entries of the vectors $\{\mathbf{q}_{ik} \circ \mathbf{u}_{ji} \circ \mathbf{v}_{kj}\}_{0 \leq i,j,k < n}$ only in terms of the entries of \mathbf{A} and its scaling values, we derive the generators for $\mathcal{I}_{\alpha,\beta,\gamma}$ as prescribed by the tautology

$$(\mathbf{q}_{00} \circ \mathbf{u}_{00} \circ \mathbf{v}_{00}) \circ (\mathbf{q}_{01} \circ \mathbf{u}_{10} \circ \mathbf{v}_{11}) \circ (\mathbf{q}_{11} \circ \mathbf{u}_{01} \circ \mathbf{v}_{10}) \circ (\mathbf{q}_{10} \circ \mathbf{u}_{11} \circ \mathbf{v}_{01}) -$$

$$(\mathbf{q}_{01} \circ \mathbf{u}_{00} \circ \mathbf{v}_{10}) \circ (\mathbf{q}_{00} \circ \mathbf{u}_{10} \circ \mathbf{v}_{01}) \circ (\mathbf{q}_{10} \circ \mathbf{u}_{01} \circ \mathbf{v}_{00}) \circ (\mathbf{q}_{11} \circ \mathbf{u}_{11} \circ \mathbf{v}_{11}) = \mathbf{0}_{2 \times 1}$$

which results in third order hypermatrix analog of $2 \times 2 \times 2$ characteristic polynomial

$$\begin{pmatrix} a_{001}a_{010}a_{100}\alpha_{11}^2\beta_{11}^2\gamma_{11}^2 - a_{011}a_{101}a_{110}\alpha_{01}^2\beta_{01}^2\gamma_{01}^2 + a_{000}a_{011}a_{101}a_{110} - a_{001}a_{010}a_{100}a_{111} \\ a_{001}a_{010}a_{100}\alpha_{01}^2\beta_{01}^2\gamma_{01}^2 - a_{011}a_{101}a_{110}\alpha_{00}^2\beta_{00}^2\gamma_{00}^2 + a_{000}a_{011}a_{101}a_{110} - a_{001}a_{010}a_{100}a_{111} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first elimination method therefore yield hypermatrix analog of characteristic polynomials as well as an expression of the hyperdeterminants for hypermatrices of size $2^m \times 2^m \times 2^m$ which arises as Kronecker products of $2 \times 2 \times 2$ hypermatrices. The first method is considerably more difficult to extend to arbitrary hypermatrices of size $n \times n \times n$ because of the lack of Vandermonde for $n > 2$. However, the second elimination method extend to hypermatrices of size $n \times n \times n$. Similarly to the matrix case, the

second elimination method based on a hypermatrix generalization of Parseval's identity. We eliminate the variables associated with the scaling values by considering the sequence of hypermatrix defined the recurrence relations

$$\mathbf{G}_0 = \mathbf{\Delta}_2, \quad \mathbf{G}_{k+1} = \text{Prod}_{\mathbf{G}_k}(\mathbf{Q}, \mathbf{U}^{T^2}, \mathbf{V}^T),$$

in conjunction with the constraints

$$\left\{ \mathbf{A} = \text{Prod}_{\mathbf{G}_k} \left(\text{Prod}(\mathbf{Q}, \mathbf{D}_0, \mathbf{D}_0^T), [\text{Prod}(\mathbf{U}, \mathbf{D}_2, \mathbf{D}_2^T)]^{T^2}, [\text{Prod}(\mathbf{V}, \mathbf{D}_1, \mathbf{D}_1^T)]^T \right) \right\}_{k < n}$$

Finally using Cramer's rule we express the the scaling variables as rational functions of the entries of the $\mathbf{Q}, \mathbf{U}, \mathbf{V}$ and \mathbf{A} . As illustration for the second elimination method we consider the case of a $2 \times 2 \times 2$ hypermatrix \mathbf{A} such that $\mathbf{A}^T = \mathbf{A}$, which admit a spectral decomposition where

$$\mathbf{D}_0 = \mathbf{D}_1 = \mathbf{D}_2 \text{ and } \mathbf{Q} = \mathbf{U} = \mathbf{V}.$$

Using Cramer's rule to isolate the scaling variables from the constraints

$$\left\{ \mathbf{A} = \text{Prod}_{\mathbf{G}_k} \left(\text{Prod}(\mathbf{Q}, \mathbf{D}_0, \mathbf{D}_0^T), [\text{Prod}(\mathbf{U}, \mathbf{D}_2, \mathbf{D}_2^T)]^{T^2}, [\text{Prod}(\mathbf{V}, \mathbf{D}_1, \mathbf{D}_1^T)]^T \right) \right\}_{k < 2}$$

we derive the elimination ideal $\mathcal{I}_{\mathbf{Q}}$ from the rational identities prescribed by the tautologies :

$$\begin{cases} (\lambda_{km}^4 \lambda_{kp}^2)^3 &= \left[(\lambda_{km}^2)^3 \right]^2 \left[(\lambda_{kp}^2)^3 \right] \\ (\lambda_{km}^2 \lambda_{kn}^2 \lambda_{kp}^2)^3 &= (\lambda_{km}^2)^3 (\lambda_{kn}^2)^3 (\lambda_{kp}^2)^3 \end{cases}.$$

Unlike the first elimination method, the second elimination method has no restriction on the size of the hypermatrices. Furthermore, for the general purpose of approximating the spectrum of arbitrary hypermatrices subsequent methods discussed here construct approximation arbitrary size hypermatrices from spectral decomposition of the $2 \times 2 \times 2$ minors.

6. APPROXIMATING THE SPECTRAL DECOMPOSITIONS OF MATRICES AND HYPERMATRICES.

We describe here a recursive construction for approximating the spectral decomposition of matrices and hypermatrices based on a refinement of the proof of theorem 2. We start by discussing the matrix case and subsequently briefly discuss how the arguments are extended to hypermatrices. For some $n \times n$ matrix \mathbf{A} with complex entries, let \mathbf{A}_τ denote the matrix minor constructed as follows

$$\mathbf{A}_\tau = \mathbf{A} \circ \left[\frac{1}{n-2} \left(\left(\sum_{0 \leq i \neq \tau < n} \mathbf{e}_i \right) \cdot \left(\sum_{0 \leq j \neq \tau < n} \mathbf{e}_j \right)^T - \sum_{0 \leq k \neq \tau < n} \mathbf{e}_k \cdot \mathbf{e}_k^T \right) + \frac{1}{n-1} \sum_{0 \leq k \neq \tau < n} \mathbf{e}_k \cdot \mathbf{e}_k^T \right].$$

The minors are constructed as to obtain the identity

$$\mathbf{A} = \sum_{0 \leq \tau < n} \mathbf{A}_\tau.$$

We further assume for the sake of the argument that the spectral decomposition of the matrix minors $\{\mathbf{A}_\tau\}_{0 \leq \tau < n}$ are known and given by

$$\mathbf{A}_\tau = \left(\mathbf{U}^{[\tau]} \cdot \text{diag} \left\{ \boldsymbol{\mu}^{[\tau]} \right\} \right) \cdot \left(\mathbf{V}^{[\tau]} \cdot \text{diag} \left\{ \boldsymbol{\nu}^{[\tau]} \right\} \right)^\dagger, \quad \mathbf{I}_n - \mathbf{e}_\tau \cdot \mathbf{e}_\tau^T = \mathbf{U}^{[\tau]} \cdot \left(\mathbf{V}^{[\tau]} \right)^\dagger.$$

By concatenating the spectral decomposition of the matrices $\{\mathbf{A}_\tau\}_{0 \leq \tau < n}$ as was done in the proof of 2, we obtain that for all $0 \leq i_0, i_1 < n$

$$a_{i_0, i_1} = \sum_{0 \leq \tau, t < n} \left[\left(\sqrt{n-1} \mu_t^{[\tau]} \right) \left(\frac{u_{i_0, t}^{[\tau]}}{\sqrt{n-1}} \right) \right] \overline{\left[\left(\sqrt{n-1} \nu_t^{[\tau]} \right) \left(\frac{v_{i_1, t}^{[\tau]}}{\sqrt{n-1}} \right) \right]}$$

$$\delta_{i_0, i_1} = \sum_{0 \leq \tau, t < n} \left(\frac{u_{i_0, t}^{[\tau]}}{\sqrt{n-1}} \right) \overline{\left(\frac{v_{i_1, t}^{[\tau]}}{\sqrt{n-1}} \right)}$$

As was done in the proof of theorem 2 we construct the spectral decomposition of a larger $n^2 \times n^2$ matrix (by concatenating the spectral decomposition of the matrices $\{\mathbf{A}_\tau\}_{0 \leq \tau < n}$) which admits \mathbf{A} as a sub-matrix and expressed as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}_{01} \\ \mathbf{B}_{10} & \mathbf{B}_{11} \end{pmatrix} = (\mathbf{U} \cdot \text{diag} \{\boldsymbol{\mu}\}) \cdot (\mathbf{V} \cdot \text{diag} \{\boldsymbol{\nu}\})^\dagger, \quad \mathbf{I}_{n^2} = \mathbf{U} \cdot \mathbf{V}^\dagger.$$

Following this construction, we discuss two approximation techniques which enable a recursive approximation for the spectrum of the original matrix. The first method is the inflation approximation technique. It iteratively modifies the spectrum of the larger $n^2 \times n^2$ matrices, in order to approximate the spectrum of the smaller $n \times n$ matrix. We start from the matrices \mathbf{U} and \mathbf{V} iteratively attempts to converge via gradient descent to matrices

$$\begin{pmatrix} \mathbf{U}' & \mathbf{0}_{n \times n(n-1)} \\ \mathbf{0}_{n(n-1) \times n} & \mathbf{Q} \end{pmatrix}, \text{ and } \begin{pmatrix} \mathbf{V}' & \mathbf{0}_{n \times n(n-1)} \\ \mathbf{0}_{n(n-1) \times n} & (\mathbf{Q}^{-1})^\dagger \end{pmatrix}$$

for which we have

$$\mathbf{I}_{n^2} = \begin{pmatrix} \mathbf{U}' & \mathbf{0}_{n \times n(n-1)} \\ \mathbf{0}_{n(n-1) \times n} & \mathbf{Q} \end{pmatrix} \cdot \begin{pmatrix} (\mathbf{V}')^\dagger & \mathbf{0}_{n \times n(n-1)} \\ \mathbf{0}_{n(n-1) \times n} & \mathbf{Q}^{-1} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{A} & \mathbf{0}_{n \times n(n-1)} \\ \mathbf{0}_{n(n-1) \times n} & \mathbf{I}_{n(n-1)} \end{pmatrix} = \begin{pmatrix} \mathbf{U}' \cdot \text{diag} \{\boldsymbol{\mu}'\} & \mathbf{0}_{n \times n(n-1)} \\ \mathbf{0}_{n(n-1) \times n} & \mathbf{Q} \end{pmatrix} \cdot \begin{pmatrix} (\mathbf{V}' \cdot \text{diag} \{\boldsymbol{\nu}'\})^\dagger & \mathbf{0}_{n \times n(n-1)} \\ \mathbf{0}_{n(n-1) \times n} & \mathbf{Q}^{-1} \end{pmatrix}.$$

We therefore derive from the resulting expansion a spectral decomposition for \mathbf{A} expressed by

$$\mathbf{A} = (\mathbf{U}' \cdot \text{diag} \{\boldsymbol{\mu}'\}) \cdot (\mathbf{V}' \cdot \text{diag} \{\boldsymbol{\nu}'\})^\dagger \quad (6.1)$$

In contrast for large enough sizes we devise instead a second approximation technique referred to as the truncation method. This method starts from the spectral decomposition of the larger $n^2 \times n^2$ matrix expressed by

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}_{01} \\ \mathbf{B}_{10} & \mathbf{B}_{11} \end{pmatrix} = (\mathbf{U} \cdot \text{diag} \{\boldsymbol{\mu}\}) \cdot (\mathbf{V} \cdot \text{diag} \{\boldsymbol{\nu}\})^\dagger, \quad \mathbf{I}_{n^2} = \mathbf{U} \cdot \mathbf{V}^\dagger. \quad (6.2)$$

where the vectors $\{\mathbf{u}_i\}_{0 \leq i < n^2}$ and $\{\overline{\mathbf{v}}_j\}_{0 \leq j < n^2}$ denote n^2 -dimensional column vectors of the matrices \mathbf{U} and $\overline{\mathbf{V}}$ respectively. The approximation is therefore obtained by truncating the vectors in the matrix to obtain

$$\begin{pmatrix} \tilde{\mathbf{A}} & \mathbf{0}_{n \times (n^2-n)} \\ \mathbf{0}_{(n^2-n) \times n} & \mathbf{0}_{(n^2-n) \times (n^2-n)} \end{pmatrix} = \sum_{0 \leq k < n} \left(\mu_k \mathbf{u}_k \circ \begin{pmatrix} \mathbf{1}_{n \times 1} \\ \mathbf{0}_{(n^2-n) \times 1} \end{pmatrix} \right) \cdot \left(\nu_k \mathbf{v}_k \circ \begin{pmatrix} \mathbf{1}_{n \times 1} \\ \mathbf{0}_{(n^2-n) \times 1} \end{pmatrix} \right)^\dagger.$$

The total error incurred by the truncation from the original larger matrix is

$$\begin{aligned} \left\| \begin{pmatrix} \mathbf{A} & \mathbf{B}_{01} \\ \mathbf{B}_{01}^\dagger & \mathbf{B}_{11} \end{pmatrix} - \begin{pmatrix} \tilde{\mathbf{A}} & \mathbf{0}_{n \times (n^2-n)} \\ \mathbf{0}_{(n^2-n) \times n} & \mathbf{0}_{(n^2-n) \times (n^2-n)} \end{pmatrix} \right\|^2 &= \left\| \sum_{n \leq k < n^2} (\mu_k \mathbf{u}_k) \cdot (\nu_k \mathbf{v}_k)^\dagger \right\|^2 + \\ &\left\| \sum_{0 \leq k < n} \left(\mu_k \mathbf{u}_k \circ \begin{pmatrix} \mathbf{0}_{n \times 1} \\ \mathbf{1}_{(n^2-n) \times 1} \end{pmatrix} \right) \cdot \left(\nu_k \mathbf{v}_k \circ \begin{pmatrix} \mathbf{0}_{n \times 1} \\ \mathbf{1}_{(n^2-n) \times 1} \end{pmatrix} \right)^\dagger \right\|^2 \end{aligned}$$

which upper bounds the truncation error of the spectral approximation. We note that the approximation error may be further reduced by the use of iterative procedures based on the polynomial constraints which generate the elimination ideals. The recursive approximation scheme presented here allows us to build up an approximation for the spectral decomposition of an $n \times n$ matrix starting from the spectral decomposition of its $\binom{n}{2}$ matrix minors of size 2×2 all the way up to the n matrix minors of size $(n-1) \times (n-1)$ from which we deduce the sought after approximation of the spectral decomposition. The approximation algorithms described here straightforwardly extend to hypermatrices of all orders.

7. UNITARY AND HERMITIAN HYPERMATRICES.

We describe the spectral decomposition of even order Hermitian hypermatrices. We also introduce here unitary hypermatrices and show how they can be used to extend to hypermatrices the self adjoint argument for establishing the existence of real solutions to spectral constraints. The discussion here relates the spectral decomposition to the tensor eigenvalue first defined by Lim [Lim05] and Qi [Qi05]. An even order hypermatrix \mathbf{A} is said to be Hermitian if $\overline{\mathbf{A}^T} = \mathbf{A}$. The general spectral decomposition of fourth order hypermatrices is therefore expressed by

$$\begin{aligned} \mathbf{A} = \text{Prod} \left(\text{Prod}(\mathbf{Q}, \mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Lambda}_3), \overline{\text{Prod}(\mathbf{U}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3)^{T^3}}, \right. \\ \left. \text{Prod}(\mathbf{V}, \mathbf{\Theta}_1, \mathbf{\Theta}_2, \mathbf{\Theta}_3)^{T^2}, \overline{\text{Prod}(\mathbf{W}, \mathbf{\Xi}_1, \mathbf{\Xi}_2, \mathbf{\Xi}_3)^T} \right) \end{aligned}$$

and

$$\left[\text{Prod}(\mathbf{Q}, \overline{\mathbf{U}^{T^3}}, \mathbf{V}^{T^2}, \overline{\mathbf{W}^T}) \right]_{i_0, i_1, i_2, i_3} = \begin{cases} 1 & \text{if } i_0 = i_1 = i_2 = i_3 \\ 0 & \text{otherwise} \end{cases}$$

where the entries of the scaling hypermatrices are given by :

$$\begin{aligned} [\mathbf{\Lambda}_1]_{i_0 i_1 i_2 i_3} &:= \delta_{i_1 i_2} \lambda_{i_1 i_3}, [\mathbf{\Lambda}_2]_{i_0 i_1 i_2 i_3} := \delta_{i_1 i_3} \lambda_{i_1 i_0}, [\mathbf{\Lambda}_3]_{i_0 i_1 i_2 i_3} := \delta_{i_1 i_0} \lambda_{i_1 i_2} \\ [\mathbf{\Gamma}_1]_{i_0 i_1 i_2 i_3} &:= \delta_{i_1 i_2} \gamma_{i_1 i_3}, [\mathbf{\Gamma}_2]_{i_0 i_1 i_2 i_3} := \delta_{i_1 i_3} \gamma_{i_1 i_0}, [\mathbf{\Gamma}_3]_{i_0 i_1 i_2 i_3} := \delta_{i_1 i_0} \gamma_{i_1 i_2} \\ [\mathbf{\Theta}_1]_{i_0 i_1 i_2 i_3} &:= \delta_{i_1 i_2} \theta_{i_1 i_3}, [\mathbf{\Theta}_2]_{i_0 i_1 i_2 i_3} := \delta_{i_1 i_3} \theta_{i_1 i_0}, [\mathbf{\Theta}_3]_{i_0 i_1 i_2 i_3} := \delta_{i_1 i_0} \theta_{i_1 i_2} \end{aligned}$$

$$[\Xi_1]_{i_0 i_1 i_2 i_3} := \delta_{i_1 i_2} \xi_{i_1 i_3}, [\Xi_1]_{i_0 i_1 i_2 i_3} := \delta_{i_1 i_3} \xi_{i_1 i_0}, [\Xi_3]_{i_0 i_1 i_2 i_3} := \delta_{i_1 i_0} \xi_{i_1 i_2}$$

where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } 0 \leq i = j < n \\ 0 & \text{otherwise} \end{cases}.$$

Entry wise the constraints are expressed as

$$\begin{aligned} a_{i_0 i_1 i_2 i_3} &= \left\langle (\lambda_{i_0} \circ \lambda_{i_2} \circ \lambda_{i_3}) \circ \mathbf{q}_{i_0, i_2, i_3}, \overline{(\gamma_{i_1} \circ \gamma_{i_0} \circ \gamma_{i_3}) \circ \mathbf{u}_{i_1 i_3 i_0}}, \right. \\ &\quad \left. (\theta_{i_2} \circ \theta_{i_0} \circ \theta_{i_1}) \circ \mathbf{v}_{i_2 i_0 i_1}, \overline{(\xi_{i_3} \circ \xi_{i_1} \circ \xi_{i_2}) \circ \mathbf{w}_{i_3 i_1 i_2}} \right\rangle \\ \delta_{i_0 i_1 i_2 i_3} &= \langle \mathbf{q}_{i_0 i_2 i_3}, \overline{\mathbf{u}_{i_1 i_3 i_0}}, \mathbf{v}_{i_2 i_0 i_1}, \overline{\mathbf{w}_{i_3 i_1 i_2}} \rangle. \end{aligned}$$

In particular for unitary decomposition we have

$$\begin{aligned} a_{i_0 i_1 i_2 i_3} &= \left\langle (\lambda_{i_0} \circ \lambda_{i_2} \circ \lambda_{i_3}) \circ \mathbf{q}_{i_0 i_2 i_3}, \overline{(\gamma_{i_1} \circ \gamma_{i_0} \circ \gamma_{i_3}) \circ \mathbf{q}_{i_1 i_3 i_0}}, \right. \\ &\quad \left. (\theta_{i_2} \circ \theta_{i_0} \circ \theta_{i_1}) \circ \mathbf{q}_{i_2 i_0 i_1}, \overline{(\xi_{i_3} \circ \xi_{i_1} \circ \xi_{i_2}) \circ \mathbf{q}_{i_3 i_1 i_2}} \right\rangle \end{aligned}$$

and the unitarity constraints are entry wise expressed by

$$\delta_{i_0, i_1, i_2, i_3} = \langle \mathbf{q}_{i_0 i_2 i_3}, \overline{\mathbf{q}_{i_1 i_3 i_0}}, \mathbf{q}_{i_2 i_0 i_1}, \overline{\mathbf{q}_{i_3 i_1 i_2}} \rangle.$$

The vectors $\{\lambda_i \circ \lambda_j \circ \lambda_k, \overline{\gamma_i \circ \gamma_j \circ \gamma_k}, \theta_i \circ \theta_{i_0} \circ \theta_{i_1}, \overline{\xi_{i_3} \circ \xi_{i_1} \circ \xi_{i_2}}\}_{0 \leq k < n}$ denote the vectors collecting the scaling values of the hypermatrix \mathbf{A} . Note that the unitary decomposition described here for even order hypermatrices is analogous to spectral decomposition of Hermitian matrices expressed by

$$\mathbf{A} = (\mathbf{U} \cdot \text{diag}\{\boldsymbol{\mu}\}) \cdot \overline{(\mathbf{U} \cdot \text{diag}\{\boldsymbol{\nu}\})^T}, \quad \mathbf{U} \cdot \overline{\mathbf{U}^T} = \mathbf{I}_n$$

entry wise expressed as

$$a_{i_0 i_1} = \langle \boldsymbol{\mu} \circ \mathbf{u}_{i_0}, \overline{\boldsymbol{\nu} \circ \mathbf{u}_{i_1}} \rangle, \quad \delta_{i_0 i_1} = \langle \mathbf{u}_{i_0}, \overline{\mathbf{u}_{i_1}} \rangle$$

where the vectors $\boldsymbol{\mu}, \boldsymbol{\nu}$ correspond to the scaling vectors. Moreover, \mathbf{A} is said to admit slice invariant unitary decomposition if

$$\forall 0 \leq i < j < n, \quad \lambda_i = \lambda_j; \gamma_i = \gamma_j; \theta_i = \theta_j; \xi_i = \xi_j.$$

In the case of matrices the spectral decomposition of a Hermitian matrix is always slice invariant because the scaling vectors do not change as the index of the eigenvector entries changes.

Theorem 3. *Let \mathbf{A} denotes a Hermitian hypermatrix which admits a slice invariant unitary decomposition then it follows that the Hadamard product of the scaling vectors must be real.*

The general argument of the proof is well illustrated for hypermatrices of order 2 and 4. It will immediately be apparent how to extend the argument to arbitrary even order hypermatrices.

Proof. In the case of matrices we consider the bilinear form $\langle \mathbf{x}, \overline{\mathbf{y}} \rangle_{\mathbf{A}}$ associated with the matrix \mathbf{A} . Let the spectral decomposition of the matrix \mathbf{A} be

$$\mathbf{A} = (\mathbf{U} \cdot \text{diag}\{\boldsymbol{\mu}\}) \cdot \overline{(\mathbf{U} \cdot \text{diag}\{\boldsymbol{\nu}\})^T}, \quad \mathbf{U} \cdot \overline{\mathbf{U}^T} = \mathbf{I}_n$$

then the corresponding bilinear form can be expressed as follows

$$\langle \mathbf{x}, \bar{\mathbf{y}} \rangle_{\mathbf{A}} = \sum_{0 \leq k < n} \langle \boldsymbol{\mu} \circ \mathbf{x}, \bar{\boldsymbol{\nu}} \circ \bar{\mathbf{y}} \rangle_{\mathbf{u}_k \bar{\mathbf{u}}_k^T}$$

where \mathbf{u}_k denotes the k -th column of the unitary matrix \mathbf{Q} . The bilinear form associated with the matrix $\bar{\mathbf{A}}^T$ is therefore given by

$$\langle \mathbf{x}, \bar{\mathbf{y}} \rangle_{\bar{\mathbf{A}}^T} = \sum_{0 \leq k < n} \langle \bar{\boldsymbol{\mu}} \circ \mathbf{x}, \boldsymbol{\nu} \circ \bar{\mathbf{y}} \rangle_{\mathbf{u}_k \bar{\mathbf{u}}_k^T}.$$

By Hermiticity of \mathbf{A} we have

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n, \quad \langle \mathbf{x}, \bar{\mathbf{y}} \rangle_{\mathbf{A}} = \langle \mathbf{x}, \bar{\mathbf{y}} \rangle_{\bar{\mathbf{A}}^T} \Rightarrow \boldsymbol{\mu} \circ \bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\mu}} \circ \boldsymbol{\nu},$$

by the combinatorial Nullstellensatz argument thus deriving that the eigenvalues of \mathbf{A} must all be real. Similarly we consider the multilinear form associated with Hermitian hypermatrix \mathbf{A} which admits a scale invariant unitary decomposition. The corresponding multilinear form is expressed by

$$\langle \mathbf{x}, \bar{\mathbf{y}}, \mathbf{z}, \bar{\mathbf{t}} \rangle_{\mathbf{A}} = \sum_{0 \leq k < n} \langle (\boldsymbol{\lambda} \circ \mathbf{x}), (\bar{\boldsymbol{\gamma}} \circ \bar{\mathbf{y}}), (\boldsymbol{\theta} \circ \mathbf{z}), (\bar{\boldsymbol{\xi}} \circ \bar{\mathbf{t}}) \rangle_{\text{Prod}(\mathbf{u}_k, \bar{\mathbf{u}}_k^{T^3}, \mathbf{u}_k^{T^2}, \bar{\mathbf{u}}_k^T)}.$$

The multilinear form associated with the hypermatrix $\bar{\mathbf{A}}^T$ is therefore given by

$$\langle \mathbf{x}, \bar{\mathbf{y}}, \mathbf{z}, \bar{\mathbf{t}} \rangle_{\bar{\mathbf{A}}^T} = \sum_{0 \leq k < n} \langle (\boldsymbol{\gamma} \circ \mathbf{x}), (\bar{\boldsymbol{\theta}} \circ \bar{\mathbf{y}}), (\boldsymbol{\xi} \circ \mathbf{z}), (\bar{\boldsymbol{\lambda}} \circ \bar{\mathbf{t}}) \rangle_{\text{Prod}(\mathbf{u}_k, \bar{\mathbf{u}}_k^{T^3}, \mathbf{u}_k^{T^2}, \bar{\mathbf{u}}_k^T)}.$$

By Hermiticity we have

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t} \in \mathbb{C}^n, \quad \langle \mathbf{x}, \bar{\mathbf{y}}, \mathbf{z}, \bar{\mathbf{t}} \rangle_{\mathbf{A}} = \langle \mathbf{x}, \bar{\mathbf{y}}, \mathbf{z}, \bar{\mathbf{t}} \rangle_{\bar{\mathbf{A}}^T} \Rightarrow \boldsymbol{\lambda} \circ \bar{\boldsymbol{\gamma}} \circ \boldsymbol{\theta} \circ \bar{\boldsymbol{\xi}} = \bar{\boldsymbol{\lambda}} \circ \boldsymbol{\gamma} \circ \bar{\boldsymbol{\theta}} \circ \boldsymbol{\xi}.$$

□

Theorem 3 which extends to hypermatrices the self-adjointness argument for establishing the existence of real solutions to spectral constraints. We may also write that

$$\langle \mathbf{x}, \bar{\mathbf{x}}, \mathbf{x}, \bar{\mathbf{x}} \rangle_{\mathbf{A}} = \sum_{0 \leq k < n} \langle (\boldsymbol{\lambda} \circ \mathbf{x}), (\bar{\boldsymbol{\gamma}} \circ \bar{\mathbf{x}}), (\boldsymbol{\theta} \circ \mathbf{x}), (\bar{\boldsymbol{\xi}} \circ \bar{\mathbf{x}}) \rangle_{\text{Prod}(\mathbf{u}_k, \bar{\mathbf{u}}_k^{T^3}, \mathbf{u}_k^{T^2}, \bar{\mathbf{u}}_k^T)}.$$

and therefore if we further make the simplifying assumption that for some positive real number μ

$$\forall 0 \leq i, k < n, \quad \mu \leq \min \{ \langle \mathbf{e}_i, \boldsymbol{\lambda} \rangle, \langle \mathbf{e}_i, \boldsymbol{\gamma} \rangle, \langle \mathbf{e}_i, \boldsymbol{\theta} \rangle, \langle \mathbf{e}_i, \boldsymbol{\xi} \rangle \}$$

and

$$\max \{ \langle \mathbf{e}_i, \boldsymbol{\lambda} \rangle, \langle \mathbf{e}_i, \boldsymbol{\gamma} \rangle, \langle \mathbf{e}_i, \boldsymbol{\theta} \rangle, \langle \mathbf{e}_i, \boldsymbol{\xi} \rangle \} \leq \nu$$

the entries of scaling hypermatrix therefore yield upper and lower bounds for the eigenvalues for the symmetrized hypermatrix associated with the multilinear forms $\langle \mathbf{x}, \bar{\mathbf{x}}, \mathbf{x}, \bar{\mathbf{x}} \rangle_{\mathbf{A}}$ introduced by [Lim05, Qi05]. The corresponding bounds are expressed by the inequality

$$\|\mu \mathbf{x}\|_{\ell_4}^4 \leq \langle \mathbf{x}, \bar{\mathbf{x}}, \mathbf{x}, \bar{\mathbf{x}} \rangle_{\mathbf{A}} \leq \|\nu \mathbf{x}\|_{\ell_4}^4. \quad (7.1)$$

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